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Properties of Analytic Univalent Function with Negative Coefficient

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Abstract

We introduce and study a new subclass of normalized analytic univalent function with negative coefficient.Necessary and sufficient conditions,distortion bounds, extreme points and radii of convexity for the class $ST_w^*(k, \alpha, \beta, \gamma)$ are obtained.

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Key Words : *Analytic function, Negative Coeffcient, Uniformly Starlike Function, Uniformly Convex Function*

1. Introduction

Ley A be the class of function f normalized by

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$D = \{z \in C : |z| < 1\}.$$

As usual, we denote by S the subclass of A, consisting of function which are also univalent in D. We recall here the definitions of the well-known classes of starlike function and convex function:

$$S^* = \left\{ f \in A : Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in D \right\},$$

$$S^c = \left\{ f \in A : Re\left(1 + \frac{zf'(z)}{f(z)}\right) > 0, \quad z \in D \right\}.$$

Let w be a fixed point in D and

$$A(w) = \{f \in H(D) : f(w) = f'(w) - 1 = 0\}.$$

In [23], Kanas and Ronning introduced the following classes

$$S_w = \{finA(w) : \text{ is univalent in } D\}$$

$$ST_w = \left\{ f \in A(w) : Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0, \quad z \in D \right\}, \tag{1.2}$$

$$CV_w = \left\{ f \in A(w) : 1 + Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0, \quad z \in D \right\}. \tag{1.3}$$

Later Acu and Owa [16] studied the classes extensively.

The class S_w^* is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class S_W^C is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [1] and [2] for uniformly starlike and convex functions, except that in this case the point w is fixed.

Let S_w denoted the subclass of $A(w)$ consisting of the function of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \quad (a_n \geq 0). \tag{1.4}$$

The functions $f(z)$ in S is said to be starlike functions of order β if and only if

$$Re\left\{\frac{(z-w)f'(z)}{f(z)}\right\} > \beta \quad (z \in D) \tag{1.5}$$

for some β $(0 \leq \beta < 1)$. we denote by $ST_w(\beta)$ the class of all starlike function of order β.

Similarly, a function $f(z)$ in S_w is said to be convex of order β if and only if

$$1+\left(Re\frac{(z-w)f'(z)}{f(z)}\right)>\beta \quad (z\in D) \tag{1.6}$$

for some β $(0\leq\beta<10$. We denote by $CV_w(\beta)$ the class of all convex functions of order. We note that the class $ST_0(\beta)$ and various other subclasses of $ST_w(\beta)$ have been studied rather extensively by Netanyahu [25], Acu and Owa [16], Clunie [14], Pommerenke [[7], [8]], Miller [15] Mogra et al [18], Uralegaddi and Ganigi [4],Aouf [17], and Uralegaddi and Somanatha ([5], [6]),Srivastava and Owa [13],pp.86 and Ghanim and Darus [[10], [12]].

For the function f(z) in the class S_w, we define

$$I^0f(z)=f(z),$$

$$I^1f(z)=(z-2)f'(z)+\frac{w}{z-w},$$

$$I^2f(z)=(z-w)(I^1f(z))'+\frac{2}{z-w},$$

and for $k=1,2,3,\cdots$ we can write

$$\begin{aligned} I^kf(z) &= (z-w)(I^{k-1}f(z))'+\frac{2}{z-w} \\ &= \frac{1}{z-w}+\sum_{n=1}^{\infty}n^k a_n(z-w)^n. \end{aligned} \tag{1.7}$$

The differential operator I^k studied extensively by Ghanim and Darus [[10], [12]] and in the case $w=0$ was given by Frasin and Darus [3].

With the help of the differential operator I^k, Ghanim and Darus[11] the class $ST_w(k,\beta)$ as follows:

Definition 1.1 : The function $f(z)\in S_w$ is said to be a member of the class $ST_w(k,\beta)$ if it satisfies

$$Re\left\{\frac{(z-w)(I^kf(z)'}{I^kf(z)}\right\}>\beta \tag{1.8}$$

$(k\in N_0=N\smile\{0\})$, for some β $(0\leq\beta<1)$ and for all z $(0\leq z<1)$ in D. It is easy to check that $ST_w(0,\beta)$ is the class of starlike functions of order $\beta.ST_w(0,0)$ gives the starlike functions for all D.

Definition 1.2 : For $\gamma \geq 0$, let $ST_w(k, \beta, \gamma)$ consist of function $f(z) \in S_w$ satisfy the condition

$$Re\left\{\frac{(z-w)(I^k f(z))'}{(I^k f(z))}\right\} > \gamma\left|\frac{(z-w)(I^k f(z)'}{(I^k f(z))} - 1\right| + \beta. \quad (1.9)$$

Definition 1.3 : For $0 \leq \alpha < 1, \gamma \geq 0$, let $ST_w(k, \alpha, \beta, \gamma)$ consist of function $f(z) \in S_w$ satisfy the condition

$$Re\left\{\frac{(1-\alpha)(z-w)(I^k f(z))' + \alpha(I^k f(z))}{(I^k f(z))}\right\}$$
$$> \gamma\left|\frac{(1-\alpha)(z-w)(I^k f(z))' + \alpha(I^k f(z))}{(I^k f(z))} - 1\right| + \beta \quad (1.10)$$

Futher, letST^* denoted the subclass of ST_w consisting functions of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n, \quad (a_n \geq 0). \quad (1.11)$$

Now, let us define

$$ST_w^*(k, \alpha, \beta, \gamma) = ST_w(k, \alpha, \beta, \gamma) \frown ST_w^*.$$

In this paper we provide necessary and sufficient conditions, coefficient bounds,extreme points, radius of starlikeness and convexity, closure theorems for the functions in $ST_w^*(k, \alpha, \beta, \gamma)$.

2. Characterization

We employ the technique adopted by Aqlan et al [9] to find the coefficient estimates for the class $ST_w^*(k, \alpha, \beta, \gamma)$.

Theorem 2.1 : *$f \in ST_w^*(k, \alpha, \beta, \gamma)$, if and only if*

$$\sum_{n=1}^{\infty} n^k\{(1+\gamma-\alpha(1-\gamma))n + (\alpha(1-\gamma) - (\beta+\gamma))\}a_n \leq 1 - \beta \quad (2.1)$$

where $0 \leq \alpha < 1, 0 \leq \beta < 1, \gamma \geq 0$.

Proof : We have $f \in ST_w^*(k, \alpha, \beta, \gamma)$ if and only if the condition (1.8) is satisfied.

Upon the fact that

$$Re\{w\} > \gamma|w-1| + \beta \Leftrightarrow Re\{w(1+\gamma e^{i\theta}) - \gamma e^{i\theta}\} > \beta, \ \ -\pi \leq \theta \leq \pi.$$

Then for $0 < |z-w| = r < 1$, equation (1.8) may be written as

$$Re\left\{\frac{(1-\alpha)(z-w)(I^k f(z))' + \alpha(I^k f(z))}{(I^k f(z))}\ (1+\gamma e^{i\theta}) - \gamma e^{i\theta}\right\} > \beta$$

or equivalently

$$Re\left\{\frac{[1-\alpha)(z-w)(I^k f(z))' + \alpha(I^k f(z))](1+\gamma e^{i\theta}) - \gamma e^{i\theta} I^k f(z)}{I^k f(z)}\right\} > \beta. \tag{2.2}$$

Now, we let

$$A(z) = [(1-\alpha)(z-w)(I^k f(z))' + \alpha(I^k f(z))](1+\gamma e^{i\theta}) - \gamma e^{i\theta} I^k f(z)$$

and let

$$B(z) = I^k f(z).$$

Then (2.2) is equivalent to $|A(z)+(1-\beta)B(z)| > |A(z)-(1+\beta)B(z)|$ for $0 \le \beta < 1$. For $A(z)$ and $B(z)$ as above, we have

$$\begin{aligned}
&|A(z)+(1-\beta)B(z)| = \\
&\ge \frac{2-\beta}{|z-w|} - \sum_{n=1}^{\infty} n^k (n+1-\beta-\alpha(n-1)) a_n |(z-w)|^n \\
&-\gamma \sum_{n=1}^{\infty} n^k (n-1-\alpha(n-1)) a_n |(z-w)|^n
\end{aligned}$$

and similarly

$$\begin{aligned}
|A(z)-(1+\beta)B(z)| \quad < \quad & \frac{\beta}{|z-w|} + \sum_{n=1}^{\infty} n^k (n+1-\beta-\alpha(n-1)) a_n |(z-w)|^n \\
& +\gamma \sum_{n=1}^{\infty} n^k (n-1-\alpha(n-1)) a_n |(z-w)|^n.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|A(z)+(1-\beta)B(z)| - |A(z)-(1+\beta)B(z)| \\
&\frac{2(1-\beta)}{|z-w|} - 2\sum_{n=1}^{\infty} n^k (n-\beta-\alpha(n-1)) a_n |(z-w)|^n \\
&-2\gamma \sum_{n=1}^{\infty} n^k (n-1-\alpha(n-1)) a_n |(z-w)|^n \qquad (1) \\
&\ge \frac{2(1-\beta)}{|z-w|} - 2\sum_{n=1}^{\infty} n^k (n-\beta-\alpha(n-1)) a_n r^n \\
&-2\gamma \sum_{n=1}^{\infty} n^k (n-1-\alpha(n-1)) a_n r^n \ .
\end{aligned} \tag{2.3}$$

Letting $r \to 1$ in (2.3) we obtain

$$\sum_{n=1}^{\infty} n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}a_n \leq 1-\beta$$

which yields (2.1).

On the other hand, we must have

$$Re\left\{\frac{[(1-\alpha)(z-w)(I^k f(z))' + \alpha(I^k f(z))](1+\gamma e^{i\theta}) - \gamma e^{i\theta} I^k f(z)}{I^k f(z)}\right\} > \beta, \quad -\pi \leq \theta \leq \pi.$$

Upon choosing the values of $(z-w)$ on the positive real axis where $0 < |z-w| = r < 1$, the above inequality reduce to

$$Re\left\{\frac{(1-\beta) - \sum\limits_{n=1}^{\infty} n^k(n-\beta-\alpha(n-1))a_n r^{n+1} - \gamma e^{i\theta}\gamma \sum\limits_{n=1}^{\infty} n^k(n-1-\alpha(n-1))a_n r^n}{1-\sum\limits_{n=1}^{\infty} n^k(n-1)a_n r^{n+1}}\right\} > 0$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduce to

$$Re\left\{\frac{(1-\beta) - \sum\limits_{n=1}^{\infty} n^k(n-\beta-\alpha(n-1))a_n r^{n+1} - \gamma e^{i\theta}\gamma \sum\limits_{n=1}^{\infty} n^k(n-1-\alpha(n-1))a_n r^n}{1-\sum\limits_{n=1}^{\infty} n^k(n-1)a_n r^{n+1}}\right\} > 0$$

Letting $r \to 1$, we get the desired result.

By taking $\alpha = 0$ in Theorem 2.1, we get

Corollary 2.2 : *Let $f \in ST_w^*$. Then $f \in ST_w^*(k, 0, \beta, \gamma)$, if and only if*

$$\sum_{n=1}^{\infty} n^k[(n-\beta)+\gamma(n-1)]a_n \leq 1-\beta. \tag{2.4}$$

Taking $\alpha = \gamma = 0$ in Theorem 2.1, we get

Corollary 2.3 : *Let $f \in ST_w^*$. Then $f \in ST_w^*(k, 0, \beta, 0)$, if and only if*

$$\sum_{n=1}^{\infty} n^k(n-\beta)a_n \leq 1-\beta. \tag{2.5}$$

Theorem 2.3 : *If $f \in ST_w^*(k, \alpha, \beta, \gamma)$ then*

$$a_n \leq \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}} \tag{2.6}$$

$0 \leq \alpha < 1, 0 \leq \beta < 1, \gamma \geq 0$. *Equality holds for the function*

$$f(z) = \frac{1}{z-w} - \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}} \quad (z-w)^n.$$

Proof : Since $f \in ST_w^*(k,\alpha,\beta,\gamma)$ holds. Since

$$\sum_{n=1}^{\infty} n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}a_n \leq 1-\beta \tag{2.7}$$

we have,

$$a_n \leq \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}.$$

Clearly the function given by (2.6) is in $f \in ST_w^*(k,\alpha,\gamma)$. For this function, the result is clearly sharp.

By taking $\alpha = 0$ in Theorem 2.3, we get

Corollary 2.4 : *Let $f \in ST_w^*$. Then $f \in ST_w^*(k,0,\gamma)$, then*

$$a_n \leq \frac{1-\beta}{n^k[n-\beta)+\gamma(n-1)}.$$

Corollary 2.5 : *If $f(z) \in ST_w^*(k,\alpha,\gamma)$, then*

$$na_n \leq \frac{1-\beta}{n^{k-1}\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}} \tag{2.8}$$

$0 \leq \alpha < 1, 0 \leq \quad < 1, \gamma = 0.$

3. Distortion Theorems for $ST_w^*(k,\alpha,\beta,\gamma)$

Theorem 3.1 : *If $f \in ST_w^*(k,\alpha,\beta,\gamma)$, then*

$$\frac{1}{r} - r \leq |f(z)| \leq \frac{1}{r} + r. \tag{3.1}$$

Proof : Since $f \in ST_w^*(k,\alpha,\beta,\gamma)$, by Theorem 2.1,

$$\sum_{n=1}^{\infty} n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}a_n \leq 1-\beta.$$

Note that

$$(1-\beta)\sum_{=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}a_n \leq 1-\beta$$

and therefore

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\beta}{1-\beta} = 1.$$

Since

$$\begin{aligned} f(z) &= \frac{1}{z-2} + \sum_{n=1}^{\infty} a_n (z-w)^n, \\ |f(z)| &= \left| \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \right| \\ &\leq \frac{1}{|z-w|} + \sum_{n=1}^{\infty} a_n |(z-w)|^n \\ &\leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n \leq \frac{1}{r} + r \end{aligned}$$

and

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \right| \\ &\geq \frac{1}{|z-w|} - \sum_{n=1}^{\infty} a_n |(z-w)|^n \\ &\geq \frac{1}{r} - r \sum_{n=1}^{\infty} a_n \geq \frac{1}{r} - r. \end{aligned}$$

Thus

$$\frac{1}{r} - r \leq |f(z)| \leq \frac{1}{r} + r$$

which yields the inequality (3.1).

Theorem 3.2 : *If* $f \in ST_w^*(k, \alpha, \beta, \gamma)$, *then*

$$\frac{1}{r^2} - 1 \leq |f(z)| \leq \frac{1}{r^2} + 1. \tag{3.2}$$

Proof : We have

$$|f'(z)| = \left| \frac{1}{(z-w)^2} + \sum_{n=1}^{\infty} n a_n (z-w)^{n-1} \right| \leq \frac{1}{r^2} \sum_{n=1}^{\infty} n a_n. \tag{3.3}$$

Since, $f(z) \in ST_w^*(k, \alpha, \beta, \gamma)$, we have

$$\begin{aligned} (1-\beta) \sum_{n=1}^{\infty} n a_n &\leq \sum_{n=1}^{\infty} n^k \{(1+\gamma-\alpha(1-\gamma))n + (\alpha(1-\gamma)-(\beta+\gamma))\} a_n \\ &\leq 1-\beta. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} na_n \leq \frac{1-\beta}{1-\beta} = 1. \tag{3.4}$$

Substituting (3.4) in (3.3), we get

$$|f'(z)| \leq \frac{1}{r^2} + 1. \tag{3.5}$$

Similarly,

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{n=1}^{\infty} na_n \geq \frac{1}{r^2} - 1.$$

This completes the proof of the theorem.

4. Extreme Points for $ST_w^*(k, \alpha, \beta, \gamma)$

Theorem 4.1 : *If*

$$f_0(z) = \frac{1}{z-w}$$

and

$$f_n(z) = \frac{1}{z-w} - \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n + (\alpha(1-\gamma)-(\beta+\gamma))\}}(z-w)^n, n = 1, 2, 3, \cdots.$$

Then $f \in ST_w^*(k, \alpha, \beta, \gamma)$*, if and only if it can be represented in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where λ_n *and* $\sum\limits_{n=0}^{\infty} \lambda_n = 1$.

Proof : Suppose f can be expressed in (4.1). Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_n \left\{ \frac{1}{z-w} - \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n + (\alpha(1-\gamma)-(\beta+\gamma))\}}(z-w)^n \right\} \\ &= \frac{1}{z-w} - \sum_{n=1}^{\infty} \lambda_n \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n + (\alpha(1-\gamma)-(\beta+\gamma))\}}(z-w)^n. \end{aligned}$$

Now,

$$\sum_{n=1}^{\infty} \frac{n^k\{(1+\gamma-\alpha(1-\gamma))n + (\alpha(1-\gamma)-(\beta+\gamma))\}\lambda_n}{1-\beta} \times$$

$$\frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n + (\alpha(1-\gamma)-(\beta+\gamma))}$$

$$= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1.$$

Thus $f \in ST_w^*(k,\alpha,\beta,\gamma)$.

Conversely, suppose $f \in ST_w^*(k,\alpha,\beta,\gamma)$, then

$$a_n \leq \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}, \quad n\ 1,2,3,\cdots$$

and therefore we may set

$$\lambda_n = \frac{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}{1-\beta}, \quad n=1,2,3,\cdots$$

and

$$\lambda_0 = 1-\sum_{n=1}^{\infty}\lambda_n.$$

Then

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z), \tag{4.2}$$

and hence the proof is complete.

5. Radius of Starlikeness and Convexity

Theorem 5.1 : *If $f \in ST_w^*(k,\alpha,\beta,\gamma)$, then f is starlikeness of order α in $|z-w| < r_1$, where*

$$r_1 = \inf_{n\geq 1}\left\{\frac{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}(1-\alpha)}{(1-\beta)(n+2-\alpha)}\right\}^{\frac{1}{n+1}}. \tag{5.1}$$

Proof : It is sufficient to prove that

$$\left|\frac{(z-w)f'(z)}{f(z)}+1\right|$$

for $|z-w| < r_1$ we have

$$\left|\frac{(z-w)f'(z)}{f(z)}+1\right| = \left|\frac{-\sum\limits_{n=1}^{\infty}(n+1)a_n(z-w)^n}{\frac{1}{z-w}-\sum\limits_{n=1}^{\infty}a_n(z-w)^n}\right|$$

$$= \left|\frac{\sum\limits_{n=1}^{\infty}(n+1)a_n(z-w)^{n+1}}{1-\sum\limits_{n=1}^{\infty}a_n(z-w)^{n+1}}\right|$$

$$\leq \frac{\sum\limits_{n=1}^{\infty}(n+1)a_n|z-w|^{n+1}}{1-\sum\limits_{n=1}^{\infty}a_n|z-w|^{n+1}}. \tag{5.2}$$

Hence (5.2) holds true if

$$\frac{\sum\limits_{n=1}^{\infty}(n+1)a_n|z-w|^{n+1}}{1-\sum\limits_{n=1}^{\infty}a_n|z-w|^{n+1}} \leq 1-\alpha \tag{5.3}$$

or

$$\frac{\sum\limits_{n=1}^{\infty}(n+w-\alpha)a_n|z-w|^{n+1}}{1-\alpha} \leq 1 \tag{5.4}$$

with the aid of (2.1), (5.4) is true if

$$\frac{(n+2-\alpha)a_n|z-w|^{n+1}}{1-\alpha} \leq \frac{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}{1-\beta} \quad (n \geq 1). \tag{5.5}$$

Solving (5.5) for $|z-w|$, we obtain

$$|z-w| \leq \left\{\frac{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}(1-\alpha)}{(1-\beta)(n+2-\alpha)}\right\}^{\frac{1}{n+1}}$$

which yields the desired result.

6. Modified Hadamard Product

For functions

$$f_j(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} a_{n,j}(z-w)^n, \quad (j=1,2)$$

in the class S_w, we define the modified Hadmard product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ by

$$f_1(z) * f_2(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} a_{n,1}a_{n,2}(z-w)^n.$$

Then, we have the following theorem.

Theorem 6.1 : *If* $f_j(z) \in sT_w^*(k,\alpha,\beta,\gamma), j=1,2$. *Then* $f_1 * f_2(z) \in ST_w^*(k,\xi,\beta,\gamma)$ *where*

$$\xi = \frac{(2-\beta)\{2^k[2(1+\gamma-\alpha(1-\gamma))+(\alpha(1-\gamma)-(\beta+\gamma))]\}-2(1-\beta)^2}{(2-\beta)\{2^k[2(1+\gamma-\alpha(1-\gamma))+(\alpha(1-\gamma)-(\beta+\gamma))]\}-(1-\beta)^2}. \tag{6.1}$$

Proof : Since $f_j(z) \in ST_w^*(k,\alpha,\beta\gamma), j=1,2,$

$$\sum_{n=1}^{\infty} n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}a_{n,j} \leq (1-\beta), (j=1,2) \tag{6.2}$$

The Cauchy-chwartz inequality leads to

$$\sum_{n=1}^{\infty} \frac{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}{1-\beta}\ \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{6.3}$$

Note that we have to find the largest ξ such that

$$\sum_{n=1}^{\infty} \frac{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}{1-\xi}\ \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{6.4}$$

Therefore, in view of (6.3) and (6.4), if

$$\frac{n-\xi}{1-\xi}\sqrt{a_{n,1}a_{n,2}} \leq \frac{n-\beta}{1-\beta} \quad (n \geq 2), \tag{6.5}$$

then (6.4) is satisfied. We have, from (6.3)

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}. \tag{6.6}$$

Thus if

$$\frac{n-\xi}{1-\xi}\left[\frac{1-\beta}{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}}\right] \leq \frac{n-\beta}{1-\beta}\ (n \geq 2) \tag{6.7}$$

or, if

$$\xi \leq \frac{(n-\beta)\{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}-n(1-\beta)^2}{(n-\beta)\{n^k\{(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}-(1-\beta)^2} = \Phi(n), n \geq 2$$

then (6.3) is satisfied. Letting we see that $\Phi(n)$ is increasing on n. This implies that

$$\xi \leq \Phi(2) = \frac{(2-\beta)\{2^k[2(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-2(1-\beta)^2}{(2-\beta)\{2^k[2(1+\gamma-\alpha(1-\gamma))n+(\alpha(1-\gamma)-(\beta+\gamma))\}-(1-\beta)^2}$$

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